B.Sc. Physics Part-I, Paper-I, Group-B

Harmonic Oscillator example in Hamiltonian dynamics:

For classical harmonic oscillator, the kinetic energy is given by $T = \frac{1}{2} m\dot{x}^2$

Potential energy $V = \frac{1}{2} k x^2$, k = const

The Lagrangian of the system is given by

$$L = T - V$$

or $$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$$

The generalized momenta p_x is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

which gives $\dot{x} = \frac{p_x}{m}$

Thus, the kinetic energy T is given by

$$T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m \left(\frac{p_x}{m}\right)^2$$

or $$T = \frac{p_x^2}{2m}$$

Thus, the Hamiltonian of the system is given by

$$H = T+V = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

Hamilton's equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\text{or} \quad p_x = m\dot{x} \quad \text{——} \quad (1)$$

$$\text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\text{or} \quad \dot{p}_x = -kx \quad \text{——} \quad (2)$$

From (1) and (2)

$$m\ddot{x} = -kx$$

$$\text{or} \quad \boxed{m\ddot{x} + kx = 0}.$$

This is the equation of a harmonic oscillator.

H.W. (1) Write the Hamiltonian for a simple pendulum and obtain its equations of motion.

(2) Using Cartesian coordinates, find the Hamiltonian for a projectile of mass m moving under uniform gravity. Obtain Hamilton's equations and identify any cyclic coordinates.